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Beyond the Berry–Robnik regime: a random matrix study of tunneling effects

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Abstract
We consider the energy spectra of quantum Hamilton systems whose classical dynamics is of mixed type, i.e. regular for some initial conditions in the classical phase space and chaotic for the complementary initial conditions. In the strict semiclassical limit when the effective Planck constant $\hbar_{\mathrm{eff}}$ is sufficiently small the Berry–Robnik (BR) statistics applies, while at larger values of $\hbar_{\mathrm{eff}}$ (or smaller energies) one sees deviations from BR due to localization and tunneling effects. We derive a two-level random matrix model, which describes these effects and can be treated analytically in a closed form. The coupling between the regular and chaotic levels due to tunneling is assumed to be Gaussian distributed. This two-level model describes most of the features of matrices of large dimensions (here $N = 1000$), which we treat numerically, and is predicted to apply in mixed type systems at low energies. The proposed analytical level spacing distribution function has two parameters, the BR parameter ρ, characterizing the classical phase space, and the coupling (antenna distortion or tunneling) parameter σ between states. Localization effects so far are not included in our analysis except in the special case where ρ can be replaced by some effective ρ_{eff}.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Random matrix theory (RMT) [1–5] has proved to be an excellent model for the statistical properties of energy spectra of classically chaotic Hamiltonian systems, even if the number of degrees of freedom is small, say just two. This is certainly one of the most dramatic manifestations or signatures of classical chaos in bound quantum systems, one of the cornerstones of quantum chaos. It goes back to the original papers by Casati, Valz-Gris and Guarneri [6], where some research toward this goal with important ideas has been published, and the classic paper by Bohigas, Giannoni and Schmit [7], where the conjecture about the universality of RMT in classically fully chaotic (ergodic) systems was formulated. It has been theoretically supported by the dynamical and semiclassical theory of spectral rigidity by M V Berry [8] using the diagonal approximation in the treatments of double sums over periodic orbits. The next important step beyond this approximation has been achieved by Sieber and Richter [9], followed by the development of an expanded semiclassical theory by S Müller *et al* [10–12]. Classically integrable quantum systems show Poissonian statistics, as is well known and corroborated by extensive numerical and analytical works (see [13] and references therein), whereas in chaotic systems the spectral fluctuations obey the statistics of the Gaussian orthogonal (GOE) or unitary (GUE) ensembles (depending on whether an antiunitary symmetry exists or not, respectively [5]). Usually the well-known Wigner surmise (formula) (see equation (19)) describes well the level spacing distribution for the infinite dimensional GOE (or GUE) level spacing distribution. For higher-order level spacing distributions useful approximate closed form formulae have been derived in [14], based on the same footing.

Typical systems in nature are, however, neither integrable nor fully chaotic but somewhere in between, namely there may be a classically regular motion on invariant tori for some initial conditions in the classical phase space, and chaotic motion for the complementary initial conditions. One important class is the nearly integrable systems, i.e. only weakly perturbed integrable systems, described by the KAM scenario (for the KAM theorem, see e.g. [15]). One often studied example is the 2D billiard family introduced in [16] and [17], but there are others such as the mushroom billiard introduced by Bunimovich [18], and currently a subject of intense theoretical, numerical and experimental studies [19, 54]. In this billiard, we have exactly one invariant chaotic and one invariant regular component.

The spectral statistics in such mixed type systems rests upon the so-called principle of uniform semiclassical condensation (PUSC) of Wigner functions of eigenstates and goes back to the notion of regular and irregular eigenstates proposed by Percival [22], the theoretical foundations by Berry [23] and the theory by Berry and Robnik [24]. The research on quantum spectra of mixed type systems was initiated in [16] and [17]. A review of these ideas, foundations and results can be found in [25] (for further references see [26–33]). The best mathematical statistical measure to analyze such systems are the so-called $E(k, L)$ probabilities[3] introduced in the context of quantum chaos by Aurich, Bäcker and Steiner [34]. The theory states that the $E(k, L)$ statistics factorize in the strict semiclassical limit of a sufficiently small effective Planck constant $\hbar_{\mathrm{eff}}$, and for the case of level spacings the formulae by Berry and Robnik [24] should apply. The factorization property is a direct consequence of statistical independence of the sequences of regular and irregular levels. Berry–Robnik (BR) theory has been verified in many different systems in the asymptotic semiclassical regime [26–37]. Recent advances in understanding the mixed type systems [38, 39] are consistent with this semiclassical limiting behavior. There has been a number of attempts in the literature

[3] $E(k, L)$ is the probability of having exactly k levels in an interval of length L—after unfolding, i.e. with mean level density equal to one. The gap probability $E(0, L)$ is directly related to the level spacing distribution $P(S) = \mathrm{d}^2E(0, S)/\mathrm{d}S^2$.

to describe correctly the level statistics in the mixed type systems, just somehow interpolating between the RMT and the Poissonian statistics, but unlike BR they are not based on sound physical grounds, and have not been confirmed in the semiclassical limit. For a review see chapter 3.2.2 of [2].

If the energy is not sufficiently large and thus $\hbar_{\mathrm{eff}}$ not sufficiently small one observes deviations from BR behavior which emerge due to both, the localization effects and the tunneling between the regular and chaotic regions in the quantum phase space of Wigner functions of eigenstates. The result is a linear behavior of $P(S)$ at small S, as predicted qualitatively already by Berry and Robnik in [24]. A fractional power-law level repulsion was discussed in [28], and this regime is observed between the linear level repulsion regime and the BR tail. The main reason for the fractional power-law level repulsion presumably lies in the localization effects. For a qualitative overall picture see [40].

The purpose of this paper is to offer a new random matrix model in order to generalize the BR level spacing distribution by including the tunneling effects between eigenstates. We shall not consider the localization effects, so we assume that the PUSC is satisfied, but the level splittings are affected by tunneling mechanisms, which couple the regular and chaotic levels, and also regular–regular through the intermediary of chaotic levels. The tunneling phenomena have been studied systematically in the context of quantum chaos since more than 15 years, and we are still quite far away from a complete general understanding. The most important contributions can be found in [41–52], and more recently in [38, 39]. The approach of Podolskiy and Narimanov [51] is very close to ours, but is semiclassical and thus, as will be seen, complementary to ours.

The influence of tunneling becomes essential whenever two eigenvalues from the chaotic and the regular region are degenerate. In this case, tunneling produces a quantum-mechanical mixture of the respective eigenfunctions ψ_C and ψ_R, and the degeneracy is lifted. The determination of the mixing angle from the physical properties of the system is a highly non-trivial task, and only first steps in this direction have been performed [53].

This was our motivation to mimic tunneling by means of a two-level random matrix model where the non-diagonal elements correspond to the tunneling matrix elements. Since hitherto nothing is known on the distribution of tunneling matrix elements in real systems, we assumed a Gaussian distribution. But we would like to stress that the approach of this paper holds for arbitrary distributions. The motivation to this work came from microwave experiments and simulations of mushroom billiards. The details of this work will be presented elsewhere [54], but one example will be shown to illustrate the performance of the model in a real physical system in section 5.

Tunneling leads to a mixture of integrable and chaotic states, but does not influence pairs from the integrable and chaotic sub-spectra, respectively. Since a two by two matrix cannot know where the two levels involved come from, a two-step procedure has been applied. First, we studied the case, where *all* pairs of eigenvalues interact via non-diagonal elements, also integrable–integrable and chaotic–chaotic ones. This situation holds, e.g., in microwave experiments, due to the presence of the measuring antenna. The *tunneling* distribution functions are then obtained by a weighted mean of the *antenna* distorted distribution functions and the undistorted BR distribution, where the weight factors just take care of the number of integrable–chaotic pairs on one hand, and integrable–integrable and chaotic–chaotic pairs on the other hand, respectively.

Following the recent advances in non-Gaussian and non-normal random matrices in [55, 56] and [57] we try to verify to what extent the properties of our matrix ensemble are structurally stable, i.e. robust against variations of the model properties such as the statistics of the matrix elements. Finally, we shall apply our theoretical results to describe the level spacing distribution for a certain configuration of the mushroom billiard, both for experimental data for microwave cavities and from the numerical data.

2. The distorted Berry–Robnik level spacing distribution

Consider 2×2 symmetric matrices $A = (A_{ij})$, where $i, j = 1$ or 2. In the present context only the difference between eigenvalues is of relevance. Without loss of generality (see [55]), we may thus assume that the trace of A vanishes, i.e.

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \tag{1}$$

where a and b are real. For the eigenvalues of A it follows

$$\lambda_{1,2} = \pm\sqrt{a^2 + b^2}, \tag{2}$$

and the level spacing distribution is given by

$$P(S) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}a\,\mathrm{d}b\,\delta(S - 2\sqrt{a^2 + b^2})g_a(a)g_b(b), \tag{3}$$

where $\delta(x)$ is the Dirac delta function, $g_a(a)$ and $g_b(b)$ are the normalized probability densities of the diagonal and off-diagonal matrix elements a and b, respectively.

Introducing polar coordinates

$$a = r\cos\varphi, \qquad b = r\sin\varphi \tag{4}$$

where $r \in [0, \infty)$ and $0 \leqslant \varphi \leqslant 2\pi$, we can do the r-integration immediately and get the general formula [56]

$$P(S) = \frac{S}{4}\int_0^{2\pi} \mathrm{d}\varphi\, g_a\left(\frac{S}{2}\cos\varphi\right) g_b\left(\frac{S}{2}\sin\varphi\right). \tag{5}$$

By construction this distribution is automatically normalized, $\langle 1\rangle = 1$, but in general this is not true for the first moment, $\langle S\rangle \neq 1$. If $g_a(x)$ and $g_b(x)$ are regular and nonzero at $x = 0$, then the integrand at $S = 0$ is just a nonzero number equal to $g_a(0)g_b(0)$, and we get for small S

$$P(S) \approx \frac{\pi S}{2} g_a(0)g_b(0). \tag{6}$$

Thus in case where both g_a and g_b are regular and nonzero at $S = 0$, we always have linear level repulsion. This linear level repulsion law is very robust, it depends only on the regularity properties of the distribution functions of the matrix elements at zero value. For regular distribution functions, $g_{a,b}(x)$, higher order corrections in S to this formula can be obtained from Taylor expansions of $g_a(x)$ and $g_b(x)$ around $x = 0$. If either $g_a(x)$ or $g_b(x)$ is zero at $x = 0$, the level repulsion is no longer linear, but of higher order, e.g. quadratic or even cubic, depending on the behavior of $g_a(x)$ and $g_b(x)$ at $x = 0$. We also see that the level repulsion is not linear in S, if $g_a(0)$ and $g_b(0)$ do not exist, since the distributions $g_a(x)$ and $g_b(x)$ are singular at $x = 0$. In such a case we obtain a fractional power-law level repulsion discovered and studied in detail in [28]. Indeed, we shall analyze such an important case in another paper, following a preliminary study in [56].

Equation (5) holds for arbitrary distributions of the diagonal and off-diagonal matrix elements of A. Let us now assume that in the absence of off-diagonal elements the level spacing distribution of A is given by the BR expression denoted by $P_{\mathrm{BR}}(S)$, which is normalized, $\langle 1\rangle_{\mathrm{BR}} = 1$ and has unit mean level spacing, $\langle S\rangle_{\mathrm{BR}} = 1$. Since the distance between the two eigenvalues of A in the absence of off-diagonal elements is $2|a|$, we find that the a distribution leading to the BR distribution must be chosen as

$$g_{\mathrm{BR}}(a) = P_{\mathrm{BR}}(2|a|). \tag{7}$$

In the presence of off-diagonal matrix elements b the BR level distance distribution is distorted. Let us first assume that in the physical system there are off-diagonal matrix elements between *all* pairs of eigenstates, so that we have the case of *all-to-all* level couplings. An example is found in the spectra obtained in a microwave experiment, due to the presence of the antenna [58][4], so for the all-to-all level coupling case we can use also the terminology antenna-distorted BR distribution, denoted by $P^{A}_{DBRN}(S)$. Here N is the dimension of the matrix and default (no index) is $N = 2$. When this distribution function is normalized to the unit mean level spacing we shall write it as $P^{An}_{DBRN}(S)$. For the sake of simplicity let us follow in our model (1) the usual practice in RMT and assume a normalized Gaussian distribution for the off-diagonal elements b. The standard deviation σ of $g_b(b)$ can be considered as a measure for the strength of the physical coupling between the coupled states. If $\sigma = 0$, we have $g_b(b) = \delta(b)$ corresponding to vanishing coupling, while at nonzero values of σ we have nonzero $b's$, representing the level coupling. It will turn out that our 2D theory, parametrized by the value of σ, is almost always in excellent agreement with the numerical simulations of the level spacing distributions for large matrices. This will be treated in section 4.

A comment on the choice of the statistics of the off-diagonal elements is appropriate at this point. For arbitrary non-Gaussian N-dimensional random matrix ensembles, it has been shown by Hackenbroich and Weidenmüller [59], using Efetov's supersymmetric techniques, that the RMT statistics of local spectral fluctuations (after the spectral unfolding) is universal for all of them, provided the limiting level distribution $N \to \infty$ is smooth and confined to a finite interval. This universality behavior generally speaking allows us to choose simply Gaussian matrix element distributions for any further purpose of large matrix simulations. However, our emsemble is different, because on the diagonal we have the BR distribution, and the off-diagonal elements are from some general ensemble. By some intuitive thinking one expects that the final result will depend mainly only on the standard deviation σ of the off-diagonal matrix elements, which is true to some extent, but nevertheless there are some deviations from the Gaussian case if we replace the off-diagonal matrix element distribution by an exponential or box (uniform) distribution with the same σ. However, the 2D analytic theory is always in agreement with the higher dimensional counterpart, and the result thus practically does not depend on the size of the matrix. This will be discussed in more detail in section 5, where we show the results of analytic and numerical studies.

We now continue with the choice of the Gaussian distribution of the off-diagonal matrix elements. For our two-dimensional model this means that we choose the ansatz

$$g_b(b) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{b^2}{2\sigma^2}\right). \tag{8}$$

The variance σ^2 serves as a parameter representing the strength of the all-to-all coupling between the eigenstates as discussed. Then, using the fact that both g_a and g_b are even, we immediately get the antenna-distorted Berry–Robnik (DBR) level spacing distribution (cf [24]):

$$P^{A}_{DBR}(S) = \frac{S}{\sigma\sqrt{2\pi}} \int_0^{\pi/2} \mathrm{d}\varphi \, P_{\mathrm{BR}}\,(S\cos\varphi) \exp\left(-\frac{S^2\sin^2\varphi}{8\sigma^2}\right) \tag{9}$$

[4] In addition all eigenvalues acquire imaginary parts due to the presence of the antenna. This effect is not considered in the present work.

One easily verifies that for $\sigma \to 0$ one comes back to $P^A_{DBR} = P_{\rm BR}$. Since we have not used special properties of the BR distribution, this formula is valid for any prescribed level spacing distribution function $P_0(S)$ instead of $P_{\rm BR}(S)$. Moreover, other g_b's can be used instead of (8). In section 3 we shall evaluate this distorted level spacing distribution by inserting the BR distribution function $P_{\rm BR}(S)$ in its explicit form into (9).

Let us briefly discuss the main features of our new distribution function $P^A_{DBR}(S)$. At a small S, if $P_{\rm BR}(0) \neq 0$, the level repulsion is linear

$$P^A_{DBR}(S) \approx \frac{S\sqrt{\pi}P_{\rm BR}(0)}{2\sigma\sqrt{2}}. \tag{10}$$

The steepness of the linear rise is inversely proportional to σ. Thus, the smaller the coupling effects, modeled by the off-diagonal elements, the steeper $P^A_{DBR}(S)$ increases with S because of the level coupling. The smaller σ, the closer is $g_b(b)$ to the delta function, i.e. to vanishing distortion. Next, for completely chaotic systems $P_{\rm BR}(S)$ reduces to the GOE level spacing distribution (19) (reminding $x \propto S$), thus $P_{\rm BR}(0) = 0$. The BR level spacing distribution $P_{\rm BR}(S)$ for small S is thus linear in S, and consequently according to (9) $P^A_{DBR}(S)$ is quadratic in S near $S = 0$. The distorted level distance distribution, therefore, has quadratic level repulsion. Exactly such a quadratic level repulsion has been found in experimental spectra from a Sinai microwave billiard [58].

In a higher dimension N, we expect that the GOE level spacing distribution maps to the GOE level spacing distribution under distortion, for σ larger than a characteristic $\sigma_c(N)$, which decreases with increasing dimension N of the random matrix at hand. For infinite dimension $\sigma_c(N \to \infty)$ probably shrinks down to zero, $\sigma_c(\infty) = 0$. Recent numerical evidence supports this expectation. Therefore, the dimension N of the matrix is also important and the dependence of the model on the dimension N has to be further explored in detail.

At the very large S/σ, the main contribution to the integral (9) comes from $\varphi \approx 0$. Then we can make the approximation $\sin\varphi \approx \varphi$, and $\cos\varphi \approx 1$, the integration interval can be extended from 0 to ∞, thereby committing only an exponentially small error, and we find, to the leading order in S/σ

$$P^A_{DBR}(S) \approx P_{\rm BR}(S), \tag{11}$$

so $P^A_{DBR}(S)$ has precisely the BR tail. We shall work out more analytical aspects of (9) in the following sections.

The distorted BR distribution defined in (9) is normalized by construction, but not its first moment. If we want to use it as a model distribution for real, experimental spectra, after the spectral unfolding it must be normalized to unit mean level spacing $\langle S\rangle = 1$. Such a normalized distribution will be denoted by $P^{An}_{DBR}(S)$. It can be easily obtained by rescaling the argument of $P^A_{DBR}(S)$ by a scale factor B_A,

$$P^{An}_{DBR}(S) = B_A P^A_{DBR}(B_A S), \qquad \text{with} \qquad B_A = \int_0^\infty x P^A_{DBR}(x)\,\mathrm{d}x. \tag{12}$$

Numerically, in our calculations, we found that B_A is typically in the range 1.01–1.20 if σ is in the range 0.05–0.30. Please note that asymptotically at large S the distribution $P^{An}_{DBR}(S)$ behaves like a stretched BR distribution, in accordance with two formulae (11) and (12). This is exactly what we shall observe in comparing this model theory with the numerical simulations of random N matrices, e.g. in figure 3.

Let us now turn to the situation that there are tunneling processes coupling only the integrable and the chaotic parts of the spectrum. This means that there are off-diagonal elements only between regular and chaotic states, but not between chaotic and chaotic nor between regular and regular ones. We introduce the BR parameters ρ_1 and $\rho_2 = 1 - \rho_1$, where

ρ_1 and ρ_2 are the relative densities of the regular and chaotic levels, respectively. Henceforth, we denote $\rho_1 = \rho$ and $\rho_2 = 1-\rho$, ρ describing the relative density of the regular states. Since the number of pairs of states taking part in the tunneling is $1-\rho_1^2-\rho_2^2 = 2\cdot\rho_1\cdot\rho_2 = 2\rho(1-\rho)$, namely the complement of the regular–regular and the chaotic–chaotic pairing, the previous distribution function (8) has to be modified into

$$g_b(b) = 2\rho(1-\rho)\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{b^2}{2\sigma^2}\right) + [1-2\rho(1-\rho)]\delta(b). \tag{13}$$

Here σ again measures the tunneling strength between regular and chaotic states, while ρ measures the relative fraction of the available regular tunneling partners. The number of pairs coupled by tunneling is described by two times the number of regular states times the number of chaotic states. With this $g_b(b)$ the resulting tunneling-distorted Berry–Robnik distribution function is now immediately obtained as

$$P_{DBR}^{T}(S) = 2\rho(1-\rho)P_{DBR}^{A}(S) + [1-2\rho(1-\rho)]P_{\mathrm{BR}}(S). \tag{14}$$

This is thus nothing but the weighted mean of the undistorted and the antenna-distorted (distorted by all-to-all couplings) Berry–Robnik distributions. For both limiting cases, $\rho = 0$ and $\rho = 1$, the original GOE and PE (Poisson ensemble) level spacing distributions, respectively, are recovered as it should be. We note that the tunneling distorted BR distribution (14) has normalized total probability to unity, but not its first moment (the mean level spacing). In order to compare the theoretical distribution with the spacing distribution from real data (experimental or numerical) the rescaling, or normalization of the first moment to one, must be performed exactly as in equation (12). Then we get the final theoretical tunneling distorted BR distribution function, by using the rescaling factor B_T

$$P_{DBR}^{Tn}(S) = B_T P_{DBR}^{T}(B_T S), \qquad \text{with} \qquad B_T = \int_0^\infty x P_{DBR}^{T}(x)\,\mathrm{d}x. \tag{15}$$

3. Analytical studies of the distorted Berry–Robnik distribution

The level spacing distribution $P(S)$ for any spectrum is always equal to the second derivative of the gap probability $E(0,S)$ in an interval of length S, $P(S) = \mathrm{d}^2E(0,S)/\mathrm{d}S^2$ [25]. The gap probability factorizes upon a statistically independent superposition of independent level sequences. From here onwards, we shall denote the gap probability by $E(S)$. The basic finding in the BR picture is the fact that $E(S)$ for the total spectrum of a mixed system is a product of all gap probabilities, whose arguments must be weighted by the classical parameters ρ_i measuring the relative volume of phase space of the regular component ($i = 1$), the largest chaotic component $i = 2$, the next largest chaotic $i = 3$, etc. We already have restricted ourselves to the simple but usually sufficient approximation of just two components, the regular one $i = 1$, with $\rho_1 \equiv \rho$, and the chaotic one $i = 2$, with $\rho_2 = 1-\rho$, such that $\rho_1+\rho_2 = 1$. Then the gap probability for the entire spectrum is $E(S) = E_1(\rho_1 S)E_2(\rho_2 S)$. (In this paper consistently the label $i = 1$ denotes the regular component and $i = 2$ the chaotic one and ρ without label means ρ_1.) Thus the BR level spacing distribution can be written in the following way

$$P_{\mathrm{BR}}(S) = \frac{\mathrm{d}^2}{\mathrm{d}S^2}E_1(\rho_1 S)E_2(\rho_2 S). \tag{16}$$

It is now useful to introduce two quantities related to $E(S)$. First $F(S)$, defined as the probability that the level spacing is larger than S. Second the cumulative level spacing distribution $W(S) = \int_0^S P(x)\,\mathrm{d}x$. The relations hold $F(S) = -\mathrm{d}E/\mathrm{d}S = 1 - W(S)$. Using this one can rewrite (16) as

$$P_{\mathrm{BR}}(S) = \rho_1^2 P_1 E_2 + 2\rho_1\rho_2 F_1 F_2 + \rho_2^2 E_1 P_2, \tag{17}$$

where the argument of each quantity with index i is $\rho_i S$, $i = 1, 2$. For the chaotic case we have to use the GOE (or GUE, GSE) results for infinite matrices, for which we have no closed form expressions. However, it is well known that the Wigner distribution, which is an exact GOE result for 2×2 matrices (here we treat only GOE, leaving GUE and GSE aside), is a good approximation to the infinite matrix case for many practical and analytical purposes. With this in mind, we have then for the Poissonian statistics in the regular case $i = 1$

$$E_1(x) = F_1(x) = P_1(x) = \mathrm{e}^{-x} \tag{18}$$

and for the entirely chaotic case, $i = 2$, the Wigner approximations (2D GOE)

$$P_2(x) = \frac{\pi x}{2}\exp\left(-\frac{\pi x^2}{4}\right), \qquad F_2(x) = 1 - W_2(x) = \exp\left(-\frac{\pi x^2}{4}\right), \tag{19}$$

and

$$E_2(x) = 1 - \mathrm{erf}\left(\frac{\sqrt{\pi}x}{2}\right) = \mathrm{erfc}\left(\frac{\sqrt{\pi}x}{2}\right), \tag{20}$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-u^2}\,\mathrm{d}u$ is the error integral and $\mathrm{erfc}(x)$ its complement, i.e. $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$. With (18)–(20) the BR level spacing distribution function (17) can be written as

$$P_{\mathrm{BR}}(x) = \mathrm{e}^{-\rho_1 x}\left\{\mathrm{e}^{-\frac{\pi\rho_2^2x^2}{4}}\left(2\rho_1\rho_2 + \frac{\pi\rho_2^3 x}{2}\right) + \rho_1^2\,\mathrm{erfc}\left(\frac{\sqrt{\pi}\rho_2 x}{2}\right)\right\}. \tag{21}$$

P_{BR} is normalized and its first moment as well, $\langle S\rangle = 1$. The second moment is always equal to $\langle S^2\rangle = 2\int_0^\infty E(x)\,\mathrm{d}x$, and is often interesting and important. For the BR distribution, it can be expressed in a closed form

$$\langle S^2\rangle_{\mathrm{BR}} = \frac{2}{\rho_1}\left\{1 - \mathrm{e}^{\frac{\rho_1^2}{\pi\rho_2^2}}\,\mathrm{erfc}\left(\frac{\rho_1}{\sqrt{\pi}\rho_2}\right)\right\}. \tag{22}$$

In the case of the Poisson statistics, $\rho_1 = 1$, $\rho_2 = 0$, it is $\langle S^2\rangle = 2$ while for the Wigner GOE case with $\rho_1 = 0$, $\rho_2 = 1$ we have $\langle S^2\rangle = 4/\pi$.

We are now going to work out good analytical approximations for the antenna distorted level spacing distribution P_{DBR}^A given in equation (9), namely for small S and for large S, by introducing (21) for P_{BR} into the general expression (9).

3.1. Small S behavior of $P_{DBR}^A(S)$

Let us expand $P_{\mathrm{BR}}(x)$ from (21) into a Taylor series at a small x, as follows

$$P_{\mathrm{BR}}(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{23}$$

Then $P_{DBR}^A(S)$ is given by the series, from equations (9) and (23),

$$P_{DBR}^A(S) = \frac{S}{\sigma\sqrt{2\pi}}\sum_{k=0}^{\infty} a_k S^k \int_0^{\pi/2} \cos^k\varphi \exp\left(-\frac{S^2\sin^2\varphi}{8\sigma^2}\right)\mathrm{d}\varphi, \tag{24}$$

which should be a good approximation for small S. The relevant integrals can be worked out

$$\int_0^{\pi/2} \cos^k \varphi \exp(-\alpha \sin^2 \varphi)\, \mathrm{d}\varphi = \frac{\sqrt{\pi}\Gamma\left(\frac{1+k}{2}\right)}{k\Gamma\left(\frac{k}{2}\right)} {}_1F_1\left(\frac{1}{2}, 1+\frac{k}{2}, -\alpha\right), \tag{25}$$

where ${}_1F_1(x)$ is the confluent hypergeometric function. Here $\alpha = S^2/8\sigma^2$. The confluent hypergeometric function can be further expressed in terms of special functions as follows:

$$k = 0 : {}_1F_1\left(\frac{1}{2}, 1, -\alpha\right) = \mathrm{e}^{-\alpha/2} I_0\left(\frac{\alpha}{2}\right), \tag{26}$$

$$k = 1 : {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -\alpha\right) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \operatorname{erf}(\sqrt{\alpha}), \tag{27}$$

$$k = 2 : {}_1F_1\left(\frac{1}{2}, 2, -\alpha\right) = \mathrm{e}^{-\alpha/2}\left[I_0\left(\frac{\alpha}{2}\right) + I_1\left(\frac{\alpha}{2}\right)\right]. \tag{28}$$

I_0 and I_1 are the modified Bessel functions of the first kind, of zero and first order, respectively. Terms $k = 3$ are complicated combinations of exponential and error functions and the complexity even increases with increasing k. The coefficients in the Taylor expansion (23) are as follows:

$$a_0 = \rho_1^2 + 2\rho_1\rho_2, \tag{29}$$

$$a_1 = -\rho_1^3 - 3\rho_1^2\rho_2 + \frac{\pi\rho_2^3}{2}, \tag{30}$$

$$a_2 = \frac{1}{2}\left(\rho_1^4 + 4\rho_1^3\rho_2 - 2\pi\rho_1\rho_2^3\right), \tag{31}$$

$$a_3 = \frac{1}{24}\left(-4\rho_1^5 - 20\rho_1^4\rho_2 + 20\pi\rho_1^2\rho_2^3 - 3\pi^2\rho_2^5\right). \tag{32}$$

Now we can expand the hypergeometric function into a Taylor series in terms of $\alpha = S^2/8\sigma^2$ and consistently work out all terms up to and including the cubic terms in S. We get, for a small S,

$$P_{DBR}^A(S) \approx \frac{S}{\sigma\sqrt{2\pi}}\left\{\frac{\pi}{2}\left(\rho_1^2 + 2\rho_1\rho_2\right) + \left(-\rho_1^3 - 3\rho_1^2\rho_2 + \frac{\pi\rho_2^3}{2}\right) S \right.$$
$$\left. + \frac{\pi}{4}\left(a_2 - \frac{a_0}{8\sigma^2}\right) S^2 + \frac{1}{3}\left(-\frac{a_1}{8\sigma^2} + 2a_3\right) S^3\right\} + O(S^5). \tag{33}$$

We see that the antenna-distorted level spacing distribution $P_{DBR}^A(S)$ is linear at $S = 0$ if $\rho_1 \neq 0$, namely

$$P_{DBR}^A(S) \approx \frac{\sqrt{\pi}}{2\sigma\sqrt{2}}\left(\rho_1^2 + 2\rho_1\rho_2\right) S. \tag{34}$$

If there is no regular part, $\rho_1 = 0$, and only the chaotic range contributes, $\rho_2 = 1$, we have quadratic level repulsion with the leading term

$$P_{DBR}^A(S) \approx \frac{\sqrt{\pi}}{2\sigma\sqrt{2}} S^2, \tag{35}$$

which, as we mentioned before, is expected due to the assumption that all pairs of levels are coupled, namely the regular and chaotic ones, the chaotic and chaotic and the regular and regular [58]; in this case obviously only chaotic–chaotic.

We have verified numerically the accuracy of approximation (33), as compared with the exact evaluation of $P^A_{DBS}(S)$, and found that the quadratic and cubic terms in the curly brackets, which stem from the expansion of the confluent hypergeometric function ${}_1F_1$, do not significantly improve the quality of the approximation for sufficiently small S. In fact, the region of good agreement (within about a few percent) with the exact formula extends up to $S \leqslant 0.4$ if only the first two terms in equation (33) are taken, and up to $S \leqslant 0.7$ if we take all the terms in (33).

3.2. Large S behavior of $P^A_{DBR}(S)$

We go back to equation (9) and calculate $P^A_{DBR}(S)$ at a large $S/\sigma \gg 1$. The main contribution to the integral comes from the interval close to $\varphi = 0$. Therefore we can make the approximations $\sin\varphi \approx \varphi$ and $\cos\varphi \approx 1 + \epsilon \equiv 1 - \varphi^2/2$. We then apply in (9) the Taylor expansion $P_{\mathrm{BR}}(S + \epsilon S) = P_{\mathrm{BR}}(S) + \epsilon S\,\mathrm{d}P_{\mathrm{BR}}/\mathrm{d}S$ and extend the upper integration limit to ∞, thereby committing only an exponentially small error. Then the integrals can be done in a closed form and we obtain

$$P^A_{DBR}(S) \approx P_{\mathrm{BR}}(S) - \frac{2\sigma^2}{S}\frac{\mathrm{d}P_{\mathrm{BR}}(S)}{\mathrm{d}S}. \tag{36}$$

The derivative $\frac{\mathrm{d}P_{\mathrm{BR}}(S)}{\mathrm{d}S}$ can be calculated explicitly from (21), namely

$$\begin{aligned}\frac{\mathrm{d}P_{\mathrm{BR}}(S)}{\mathrm{d}S} = -\frac{1}{4}\exp\left(-\rho_1 S - \frac{\pi}{4}\rho_2^2 S^2\right)\Bigg\{&12\rho_1^2\rho_2 - 2\pi\rho_2^3 + 6\pi\rho_1\rho_2^3 S + \pi^2\rho_2^5 S^2 \\ &+4\rho_1^3 \exp\left(\frac{\pi}{4}\rho_2^2 S^2\right)\mathrm{erfc}\left(\frac{\sqrt{\pi}\rho_2}{2}S\right)\Bigg\}.\end{aligned} \tag{37}$$

Thus $P^A_{DBR}(S)$ has asymptotically exactly the same tail as $P_{\mathrm{BR}}(S)$, and the first lowest correction is described in terms of the first derivative of $P_{\mathrm{BR}}(S)$. Higher order corrections increase in complexity, but can be worked out if necessary.

Using equation (36) and (14), we can also write down the asymptotic behavior of the tunneling distorted BR distribution (here $\langle S\rangle$ is not yet normalized) at large S, namely

$$P^T_{DBR}(S) \approx P_{\mathrm{BR}}(S) - [1 - 2\rho(1-\rho)]\frac{2\sigma^2}{S}\frac{\mathrm{d}P_{\mathrm{BR}}(S)}{\mathrm{d}S}. \tag{38}$$

The normalized distributions $P^{An}_{DBR}(S)$ and $P^{Tn}_{DBR}(S)$ are then calculated by the rescaling procedure as defined in equations (12) and (15). It is then exactly due to the rescaling factor B_T that the tail of $P^{Tn}_{DBR}(S)$ at large S is not exactly BR distribution, but stretched BR distribution according to equation (15), unless $B_T = 1$. The same reasoning applies to $P^{An}_{DBR}(S)$ with the rescaling factor B_A.

Finally, we should note that the Wigner approximation (19) and (20) is very useful for analytical studies, as it captures all important features of the BR and distorted BR level spacing distribution. However, in very precise calculations and analyzes, especially in numerical work, it is necessary to take the exact GOE results in the large dimension limit for P_2, F_2, E_2, to get the precise expression for the BR distribution as defined in (16) and (17), and consequently for the antenna distorted BR distribution (9). This is indeed what we did in numerical work, where we used the Padé approximations derived from [60] for the quantity $W_2(S) = 1 - F_2(S)$, from which $P_2(S)$ is obtained by differentiation and $E_2(S)$ by integration. If instead just the Wigner approximation was used, we found visible and statistically significant deviations (within one per cent or so) of the two-level formula from $P^{An}_{DBRN}(S)$ for not too large values of ρ_1, that is for $\rho_1 \leqslant 0.35$, whilst for the larger ρ_1 the Wigner approximation is good enough. This is important for the applications in the following section.

4. Simulations with random matrices

4.1. The antenna distorted Berry–Robnik distribution (all-to-all couplings)

Let us start with the situation that there are Gaussian distributed off-diagonal matrix elements for all pairs of eigenvalues. It was stated above that this corresponds to microwave cavities ('quantum billiard systems') in the presence of an antenna. This is not exactly true, since for this situation usually the matrix elements are *not* Gaussian distributed. For a delta-like scatterer, the off-diagonal matrix element coupling two states n and m is proportional to $\psi_n\psi_m$ where ψ_n and ψ_m correspond to the values of the wavefunctions of the unperturbed cavity at the antenna position (see chapter 6 of [2]). If both ψ_n and ψ_m at the antenna positions are Gaussian distributed, which is the case if both n and m are states belonging to the chaotic part of the phase space, the distribution of the product $\psi_n\psi_m$ is described by a modified Bessel function, i.e. is *non*-Gaussian! For regular eigenstates there is no universal distribution for the wavefunction amplitudes (needed at the antenna positions), with the consequence that the distribution of matrix elements coupling chaotic and regular, as well as regular and regular states, is non-universal as well. It would have been no problem to take this into account, both for the 2×2 model and the simulations. But since the main issue of the present work is the modification of the Berry–Robnik formula due to couplings, we applied a pragmatic approach and assumed Gaussian distributions with the same variance for *all* off-diagonal matrix elements describing the coupling of the chaotic–chaotic, chaotic–regular, and regular–regular states. In section 5 we study the dependence of the results on the type of the statistics of the off-diagonal matrix elements.

We introduce the normalized N-level antenna distorted BR spacing distribution function denoted by P_{DBRN}^{An}, where N is the dimension of the matrix. It is the level spacing distribution, after unfolding, of a random matrix ensemble which on the diagonal has the unfolded eigenvalues obeying the Berry–Robnik distribution (taking the exact large N limit distance distribution for the GOE part) with the given and fixed value of parameters ρ_i, $i = 1, 2, \ldots$, whereas all off-diagonal elements are Gaussian distributed (8) with the same variance σ^2. So the distribution function depends on all the BR parameters ρ_i, $i = 1, 2, \ldots$ and the coupling (or, in physical systems the tunneling) parameter σ.

If we can ignore chaotic components smaller than the dominating one, there are only two parameters left $\rho_1 = \rho$ together with $\rho_2 = 1 - \rho$ and σ. We expect the two-parameter level spacing distribution function to be an adequate description of the level spacings of real quantum dynamical systems of the mixed type.

We have done extensive numerical calculations of the said two-parameter distribution function $P_{DBRN}^{An}(S)$, with $N = 1000$. The diagonal spectrum was generated by a large block random matrix with a Poissonian block of a relative size ρ, and the remaining chaotic block of a relative size $1 - \rho$ with the GOE statistics. Unfolding was done by using the Wigner semicircular level density rule for the chaotic part. The result is the BR spectrum by definition and construction.

Then the distorted BR matrix of dimension $N = 1000$ (denoted by $DBRN$) was constructed for 1000 realizations of the same ensemble and diagonalized, so that we get one million objects in one histogram, thus achieving high statistical significance, which is manifested in small fluctuations of the histogram around the expected smooth theoretical curve. The resulting spectrum was unfolded each time by the phenomenological rule of determining the local density of levels (or local mean level spacing) by averaging over the nearest 30 levels, i.e. 15 neighbors up and down. We wish to stress that this phenomenological rule was carefully tested, using the GOE random matrices, where the optimal agreement with the exact unfolding using the Wigner semicircle rule has been sought. We should explicitly stress that although the unfolding rule with 30 neighbors is the best (for $N = 1000$), the dependence on the number of neighbors is very weak, as we tested it for 12 up to 50 neighbors. Here, again, the large N-limit has been used for the level spacing distribution of the chaotic part (GOE) of the BR distribution entering (9).

The results are presented in figures 1–3, where we compare the numerical histograms for $P_{DBRN}^{An}(S)$, $N = 1000$, with the 2D prediction of $P_{DBR}^{An}(S)$, for a variety of different values of ρ and σ. Here we must emphasize that all the plots are theoretical results for given parameters and thus no curve fitting is involved. The agreement for small variance σ^2 and all regular part fractions ρ is very good. For a reference, we also plot in each figure the BR level spacing distribution (dashed) with the same ρ to show the effect of the level coupling very clearly.

It is obvious that the two-level approximation can be expected to work properly only as long as the off-diagonal elements are small compared to the mean level spacing, i.e. if $\sigma \ll 1$. To illustrate this, figure 1 shows a comparison of the results from the simulation $P_{DBRN}^{An}(S)$ and the two-level expression (12) for $P_{DBR}^{An}(S)$ for $\sigma = 0.05$ and a number of ρ values, measuring the relative contribution of the regular states. An excellent agreement is found for all ρ values. For larger σ values the agreement between simulation and analytical curve (12) is still very good as is evident from figure 2 where $P_{DBR}^{An}(S)$ and $P_{DBRN}^{An}(S)$ are shown for $\sigma = 0.10$ and a set of ρ values. In figure 3 we compare the results for fixed $\rho = 0.5$ and varying coupling strength $\sigma = 0.01, 0.02, 0.03, 0.04$; the agreement is very good for all these weak couplings, whilst for larger coupling values, $\sigma = 0.20$ and $\sigma = 0.30$, the small deviations become visible, as expected on theoretical grounds.

We have also tested smaller matrices like $N = 30$, and found very good agreement as well.

4.2. *The tunneling distorted Berry–Robnik distribution*

To describe tunneling between the regular and chaotic phase space regions in physical systems, we assume off-diagonal matrix elements only between regular and chaotic states, but not within the regular and the chaotic block itself, respectively. Our approach is complementary to that of [51] where a semiclassical approach has been used. In the cross coupling case the two-level approximation gives an expression for the tunneling distorted Berry–Robnik formula $P_{DBR}^{Tn}(S)$, which is a weighted mean of the antenna distorted and the undistorted Berry–Robnik formula, and then normalized to $\langle S\rangle = 1$, see equations (14) and (15). Figure 4 shows results from the simulations of $P_{DBRN}^{Tn}(S)$ and the two-level expression for $P_{DBR}^{Tn}(S)$, again for $\sigma = 0.05$ and a number of ρ values. Figure 5 shows $P_{DBRN}^{Tn}(S)$ and the $P_{DBR}^{Tn}(S)$ for $\sigma = 0.10$ and a number of ρ values. In figure 6 we display the results for $\rho = 0.5$ and $\sigma = 0.01, 0.02, 0.03, 0.04$. In all these cases, the agreement between the N-level simulation and the two-level analytical formula (12) is rather encouraging. The agreement is better for larger values of ρ, i.e., if the regular part becomes dominant.

Though for $\sigma = 0.05$, the general agreement is still good, nevertheless there are significant deviations between the two-level approximation and the results from the N-dimensional simulations for small S values. In the N-dimensional simulation always a linear level repulsion is observed for small S values, while the two-level approximation starts with a nonzero value at $S = 0$. The region in S of steep linear level repulsion is indeed very small, within an exponentially small S interval $\propto \exp(-\text{const.}/\hbar_{\text{eff}})$, typical of tunneling phenomena, as can

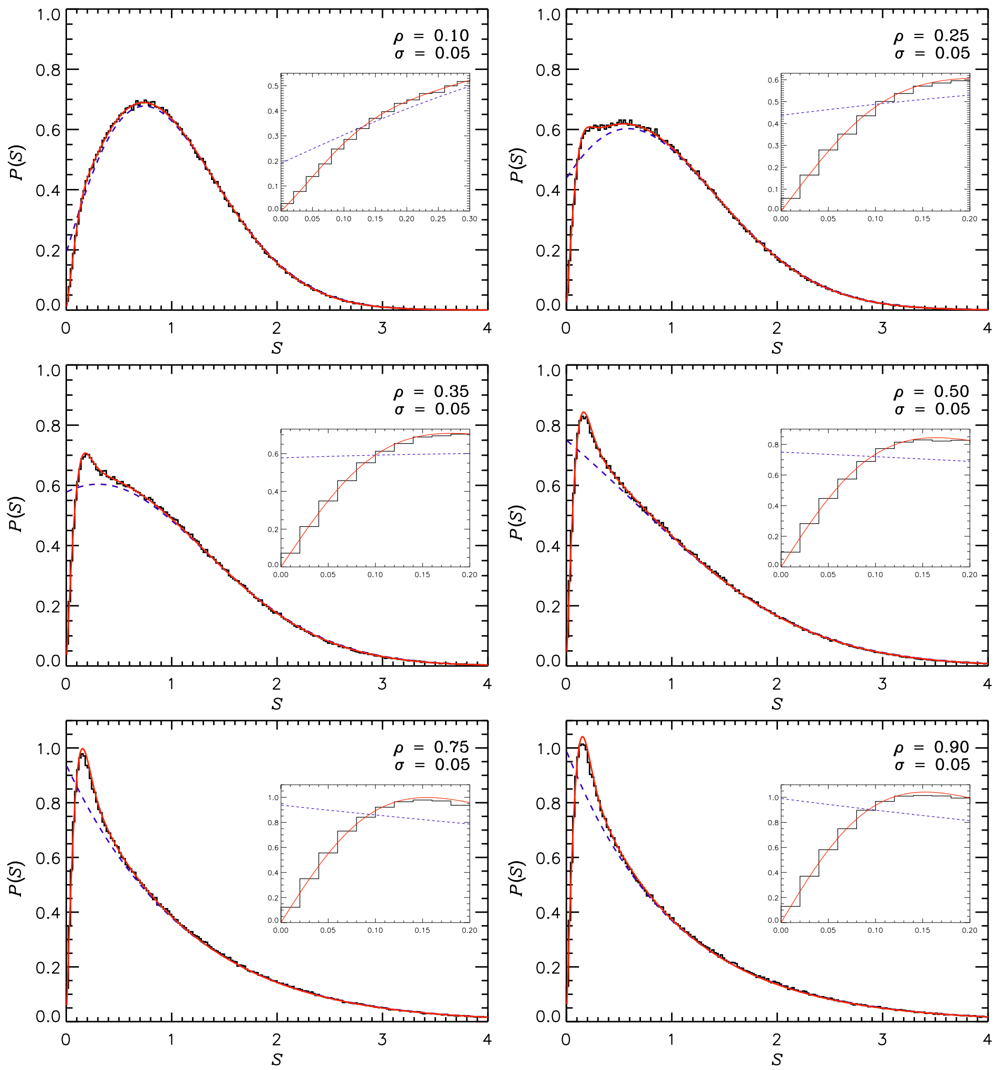

Figure 1. Numerical histograms for the antenna distorted level spacing distributions for level coupling parameter $\sigma = 0.05$ and various values of the regular fraction $\rho = 0.10, 0.25, 0.35, 0.50, 0.75, 0.90$. All eigenvalues are coupled. The full line is the analytical 2D-matrix plot from (12), the dashed curve represents the BR distribution. Numerical and analytical results are nearly indistinguishable, the BR curve deviates noticeably. Both curves are calculated using the exact evaluation of the GOE gap probability rather than the Wigner approximation. In the insets, we show the small S behavior for all three curves.

be seen in the insets of figures 4 to 6. This observation reflects the fact that in the two-level approximation the accidental degeneracies occurring generically in the regular part of the spectrum are not lifted. In the N-dimensional simulations, on the other hand, *all* degeneracies are lifted in the regular block, where there are no direct tunneling matrix elements, but second-order tunneling, which couples two regular states indirectly via one or more chaotic states. Here the two-level approximation necessarily must fail.

There is a number of alternatives to repair this defect of the 2D model. One obvious possibility is the replacement of expression (13) for the distribution of tunneling matrix

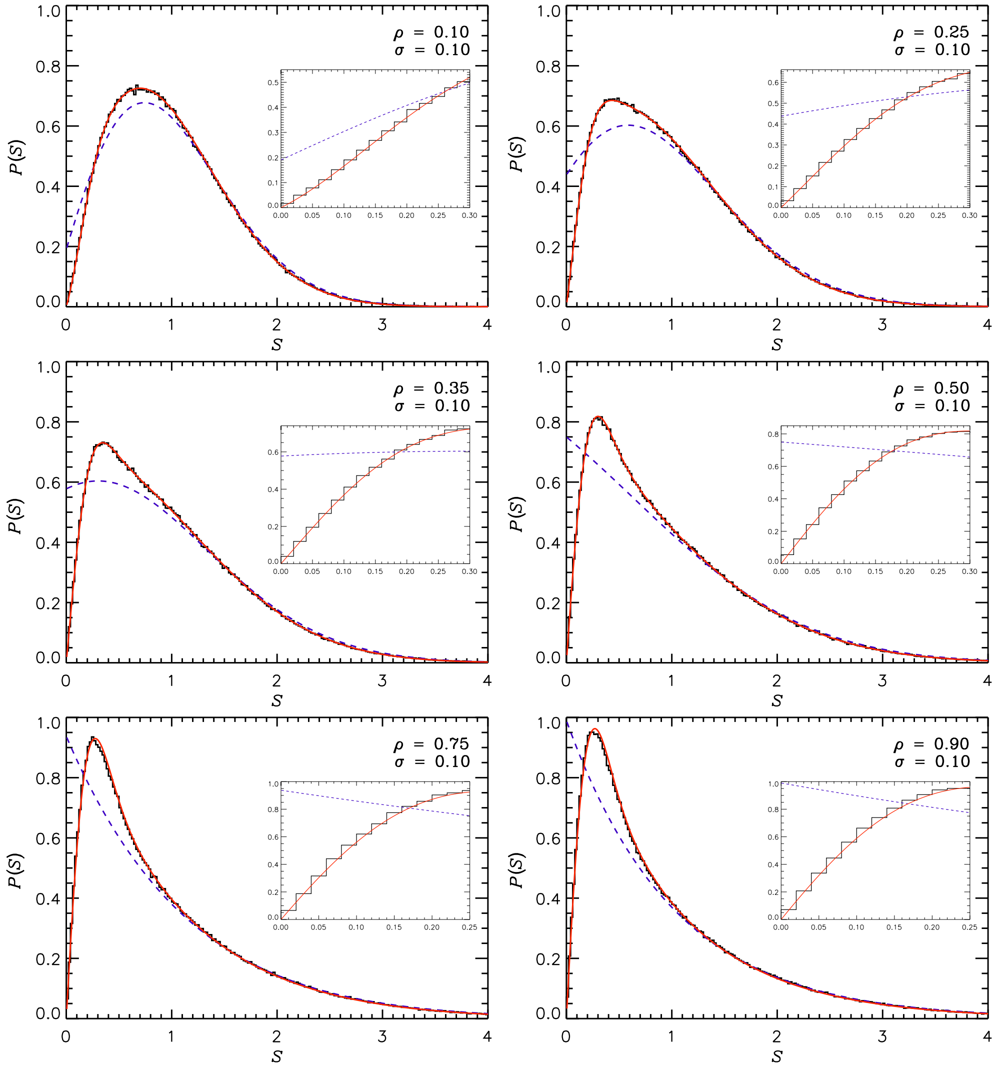

Figure 2. Same as figure 1 with the (still small) coupling parameter doubled $\sigma = 0.10$ and various values of $\rho = 0.10, 0.25, 0.35, 0.50, 0.75, 0.90$. Again all eigenvalues are coupled.

elements by

$$g_b(b) = 2\rho(1-\rho)\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{b^2}{2\sigma^2}\right) + [1-2\rho(1-\rho)]\frac{1}{\sigma_1\sqrt{2\pi}}\exp\left(-\frac{b^2}{2\sigma_1^2}\right), \qquad (39)$$

where $\sigma_1 > 0$ takes into account phenomenologically the second order splitting in the regular block. That no longer is fully regular and thus no longer is $\propto \delta(b)$. Since the splitting of the regular states is a second-order effect, σ_1 should be considered as proportional to σ^2. We have not yet been able, however, to establish an explicit relation between σ_1 and σ in this modification of the model, and therefore refrain from improving the agreement between simulations and theory by introducing a parameter, which would be ill founded. Further work to improve this somewhat unsatisfactory situation is in progress.

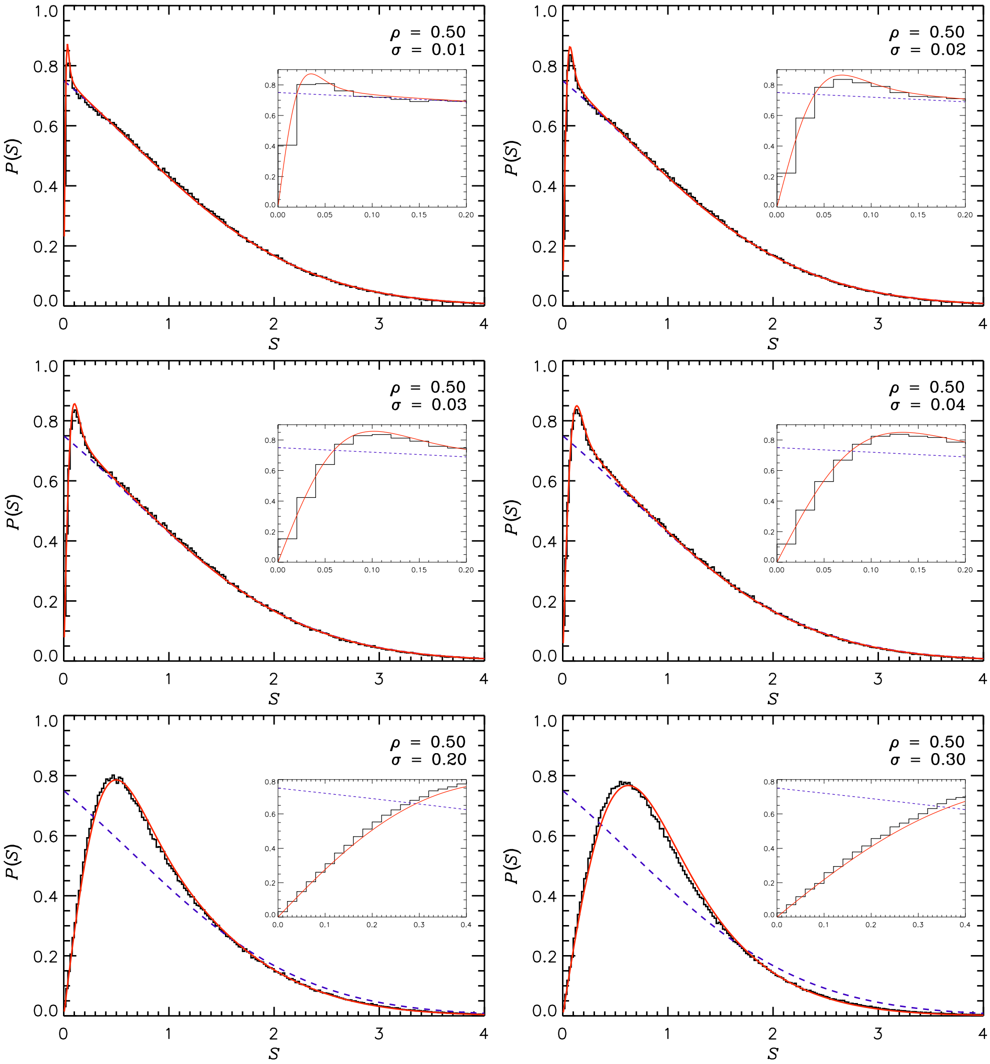

Figure 3. Same as figures 1 and 2 but for fixed $\rho = 0.5$ corresponding to equally large regular and chaotic regions and various strengths σ of the all-to-all level coupling, namely for small values $\sigma = 0.01, 0.02, 0.03, 0.04$, and for large values $\sigma = 0.2$, and $\sigma = 0.3$. All eigenvalues are coupled. At large values of σ we see small deviations of the numerical simulations from the 2D theoretical model, as expected on theoretical grounds. Please observe that at large values of $\sigma = 0.2, 0.3$ the tail of the distribution (12) (the full curve and the numerical histogram) behaves like a stretched BR distribution rather than a pure BR distribution dashed curve). At smaller values of σ this effect is not visible, because the stretch factor B_A is very close to 1.

We also mention that so far we have not treated any localization effects [49, 52], or flooding [38, 39]. Localization of chaotic states (deviation from the PUCS uniformity) would be typically modeled by suppression of tunneling between certain (basis) states. Localization near regular (stability) islands would be modeled by replacing ρ with some larger effective ρ_{eff}, whilst (partial) flooding of regular (stability) islands would be modeled by some smaller effective ρ_{eff}.

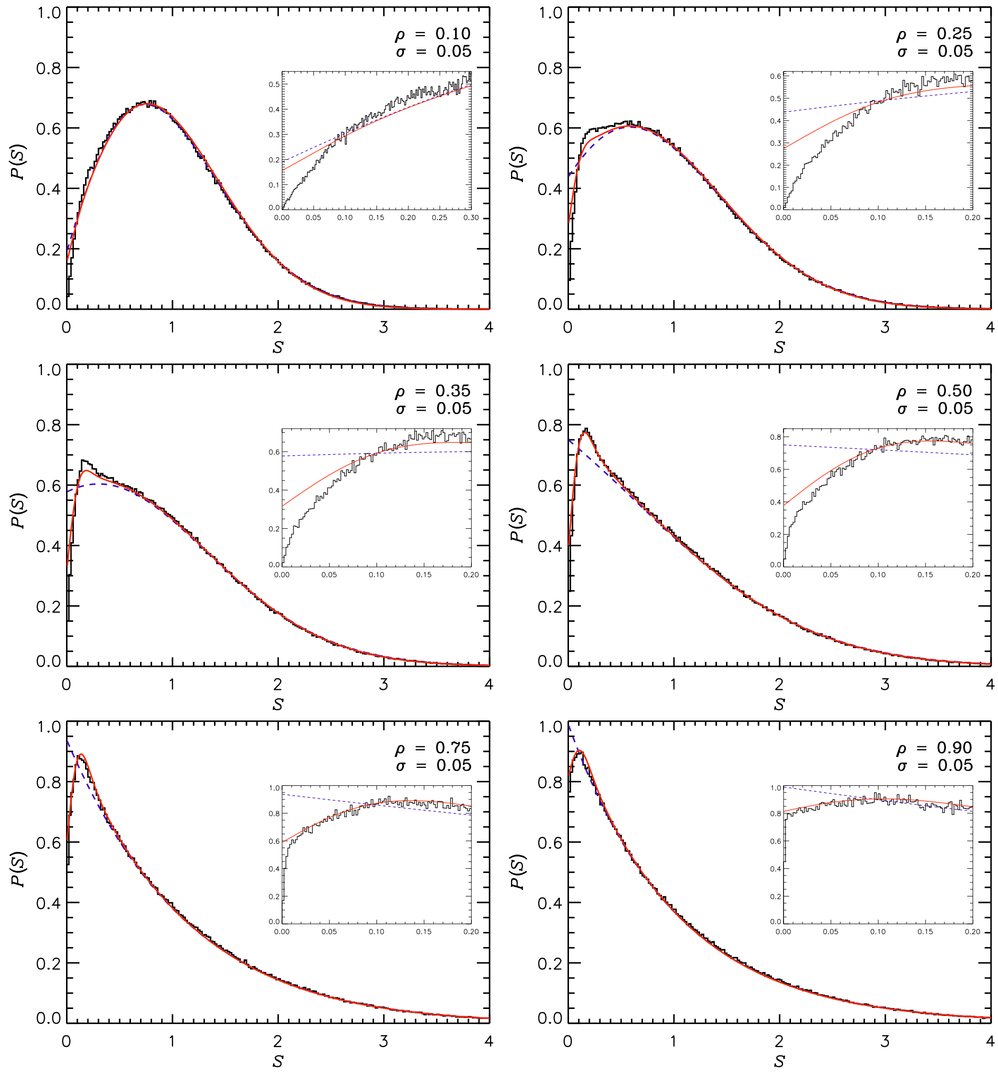

Figure 4. Comparison of the tunnelling distorted level spacing distributions for fixed coupling parameter $\sigma = 0.05$ and various sizes of the regular regions $\rho = 0.10$, 0.25, 0.35, 0.50, 0.75, 0.90. The histogram is for the numerical data P^{T}_{DBRN} and the full line for the 2D model, namely P^{Tn}_{DBR}. The dashed line is the BR distribution with the same ρ, just for comparison. Only couplings between the regular eigenvalues and the chaotic ones are allowed, no mutual coupling between the regular states themselves and none between the chaotic ones themselves. In the insets we show the small S behavior of all three curves. The tunneling formula (14–15) is in good agreement with the numerical simulations.

5. Dependence of the model on the matrix element statistics and comparison with the experimental and numerical billiard spectra

In this section we study the dependence of our model on the statistics of the off-diagonal elements, for which so far we have used only the Gaussian distribution with a given dispersion σ^2.

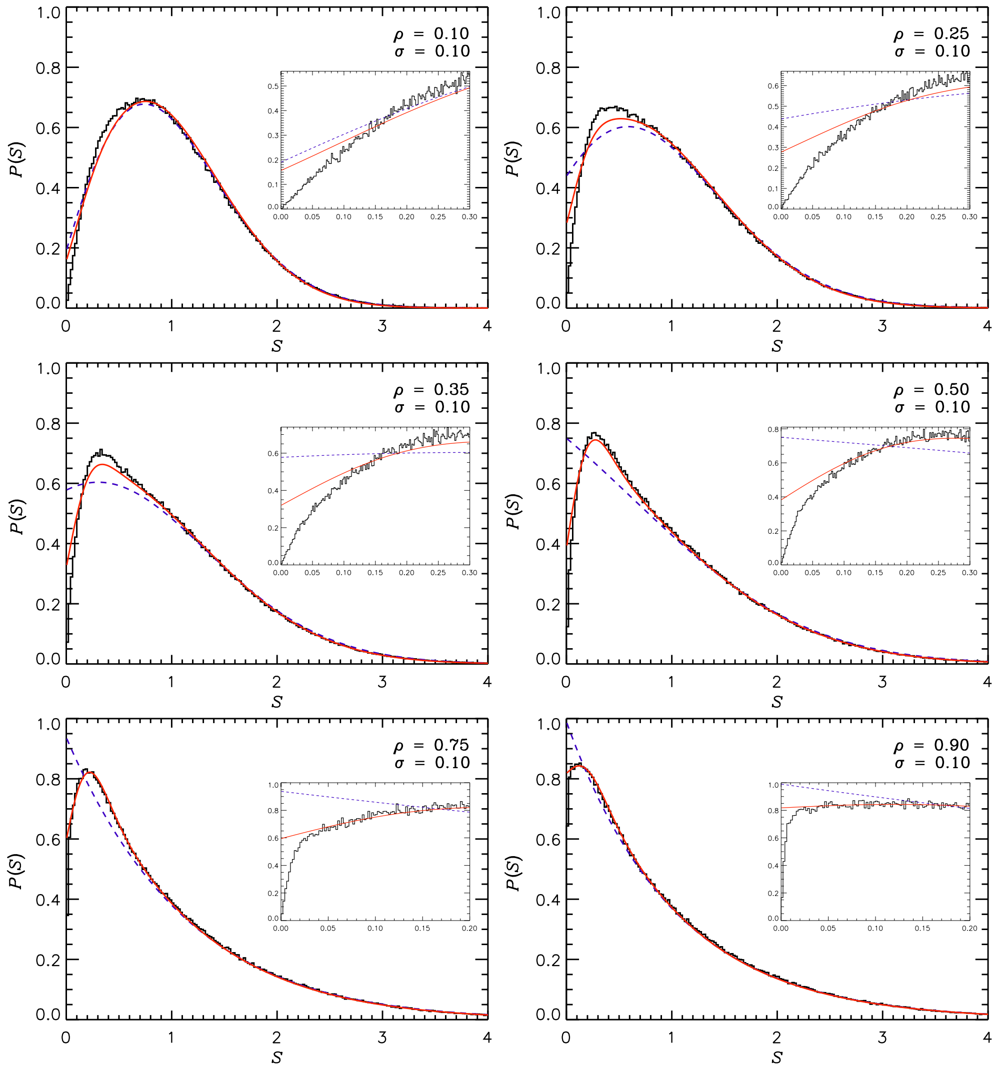

Figure 5. Same as figure 4 but twice the cross coupling (tunneling) strength $\sigma = 0.10$ and various values of the relative size of the regular regions $\rho = 0.10, 0.25, 0.35, 0.50, 0.75, 0.90$. Again only cross couplings of integrable states with chaotic ones are allowed but none within the regular or within the chaotic states themselves.

For general *non-Gaussian* random matrix ensembles, modeling fully chaotic systems, the statistics of local spectral fluctuations has been shown to obey the universal Gaussian RMT in the limit of large dimension $N \to \infty$, by Hackenbroich and Weidenmüller [59], provided the level density in this limit is smooth and confined to a finite interval. For 2D ensembles an explicit analytic theory has been performed recently by Grossmann and Robnik [55, 56] for a variety of matrix element distribution functions, including the exponential and box (uniform) distributions. Extensive numerical study [57] of such ensembles at the high dimension N confirms the Hackenbroich–Weidenmüller prediction and shows that the transition to the universal behavior is pretty fast. In cases that one of the two or both Hackenbroich–Weidenmüller conditions are not satisfied, e.g. for the Cauchy–Lorentz distribution of the

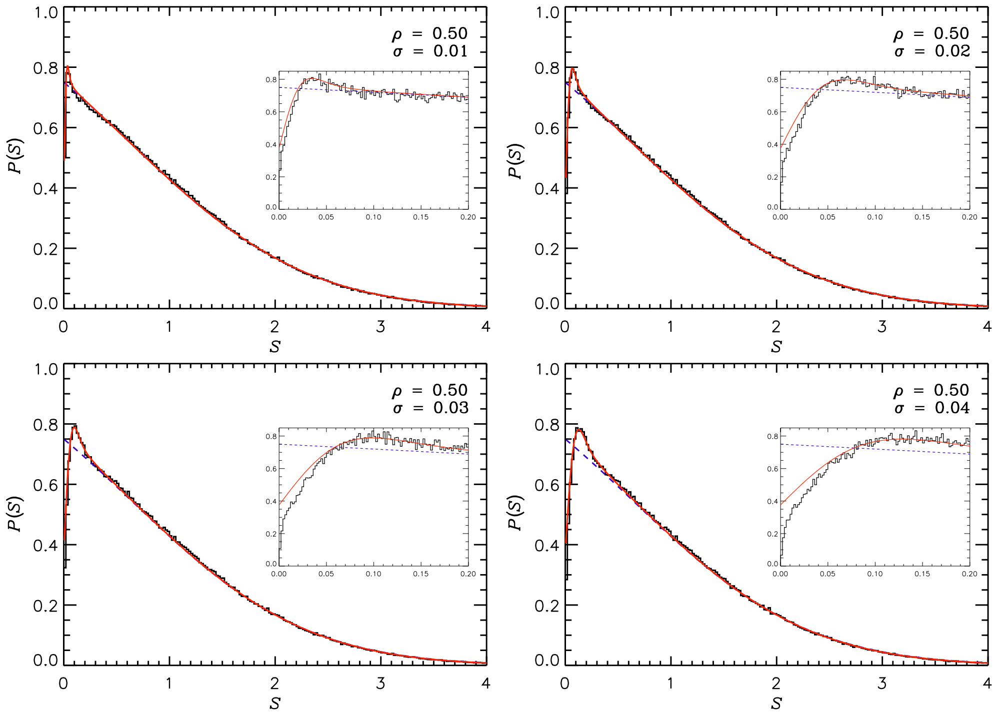

Figure 6. Same comparison of level spacing distributions as in figures 4 and 5, now for fixed $\rho = 0.5$ (equal regular and chaotic parts) but varying (small) coupling by tunneling $\sigma = 0.01, 0.02, 0.03, 0.04$. As before only cross couplings between integrable eigenvalues and chaotic ones are allowed.

matrix elements, where the level density for $N \to \infty$ does not 'live' on a finite interval, we see deviations from the GOE level spacing distribution.

It is in this spirit that we want to explore the sensitivity of the distorted BR level spacing distribution on the choice of the matrix element statistics. Thus in addition to the Gaussian case already treated, as defined in (8), we shall consider the box (uniform) distribution

$$g_b(b) = \frac{\sqrt{3}}{6\sigma} \qquad \text{if} \quad |b| \leqslant \sigma\sqrt{3}, \quad \text{and zero otherwise}, \tag{40}$$

and the exponential distribution

$$g_b(b) = \frac{\sqrt{2}}{2\sigma} \exp\left(-\frac{|b|\sqrt{2}}{\sigma}\right), \tag{41}$$

both cases with the variance σ^2.

The sensitivity on deviations from a Gaussian distribution turns out to be relatively weak, and this is clearly demonstrated in figure 7 where we plot the result for the normalized antenna distorted BR distribution $P^{An}_{DBRN}(S)$ for the Gaussian, for the exponential and for the box (uniform) distribution with the same dispersion σ^2. So these are different theoretical curves with the same σ^2, and there is no (best curve) fitting involved here. In all three cases we clearly see that the 2D theory for $P^{An}_{DBR}(S)$ of section 3 is a very good (for the Gaussian and the exponential models even excellent) approximation for the random matrix ensemble of the dimension $N = 1000$. Nevertheless, the curves differ a little bit among themselves, especially

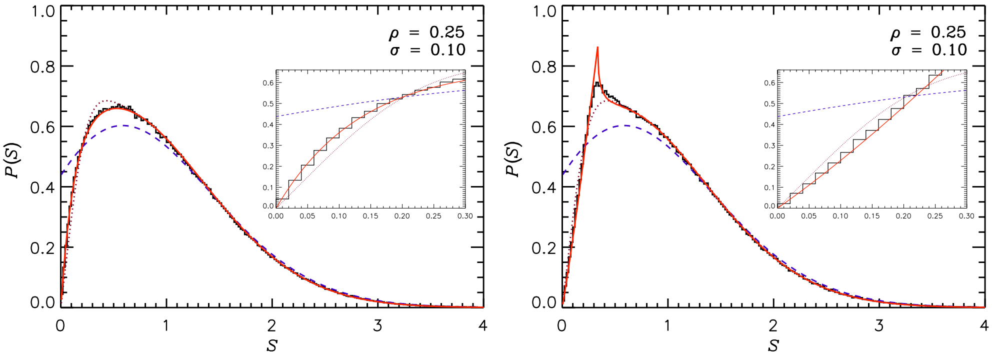

Figure 7. Comparison of various theoretical curves for the antenna distorted normalized BR level spacing distribution P_{DBRN}^{An} resulting from different distributions of the nondiagonal elements. Left: exponential distribution of the off-diagonal matrix elements: histogram from the numerical simulations with the matrices $N = 1000$ which agrees very well with the 2D formula (full curve), but they both only slightly disagree with the Gaussian model (dotted). We also show the BR distribution function with the same ρ (dashed). Right: box (uniform) distribution of the off-diagonal matrix elements: histogram from the numerical simulations with the matrices $N = 1000$ which agrees very well with the 2D formula (full curve), but they both only slightly disagree with the Gaussian model (dotted). We also show the BR distribution function with the same ρ (dashed). In the Gaussian case the $N = 1000$ result agrees very well with the 2D formula as can be seen in figure 2 (top-right).

in the central region of $0.3 \leqslant S \leqslant 0.6$, while the small S behavior (the level repulsion) and the large S behavior (the tail) are in good agreement with the Gaussian model. In fact the tail for exponential distribution is analytically the same as for the Gaussian model (36). The ratio of the slopes at $S = 0$ for the exponential and the Gaussian case is $\sqrt{\pi}$, and for the box and the Gaussian case it is $\sqrt{\pi/6}$, and remains so within 2% after rescaling of the first moment to unity, using the stretch factor B_A. The agreement between the Gaussian and other ensembles is, therefore, quite satisfactory. Indeed, it is our intuitive expectation that the statistical spectral properties of random matrix ensembles depend essentially only on the variance of the matrix elements and not on other details of the ensemble, unless they become singular in a one or another way.

Next in figure 8, we show the results for the spectra of a mushroom billiard whose geometry is defined in the insets. We see that the best-fitting Gaussian model for P_{DBR}^{An} for the experimental data works quite well, as well as the Gaussian tunneling model P_{DBR}^{Tn} in the case of the numerical billiard spectra. Similarly, the best fitting exponential model is of comparable quality. In both cases, σ is the only fitting parameter, and ρ is determined by classical dynamics. In the case of the experimental data, we find $\sigma_G = 0.187$ for the Gaussian model and $\sigma_E = 0.265$ for the exponential model, while in the case of the numerical spectra we find $\sigma_G = 0.133$ for the Gaussian model and $\sigma_E = 0.156$ for the exponential model.

For the sake of completeness, we mention briefly that in calculating the numerical spectra we have employed the expanded boundary integral method developed very recently by Veble *et al* [61]; we have used sequential levels from 180 to 280 for 81 geometrical configurations of equidistantly varying parameter h (step $\Delta h = 0.0013$) around the central position at $h = 10/19$. (For the concave corner of 270° we have locally used a denser mesh of points by factor one to four linearly increasing over an interval of length 0.4 away from the corner.) The parameter ρ for these configurations varies only by about 1.4% around the central value $\rho = 0.269$. The maximal absolute error in calculating the energy levels was about 0.004 in units of the level spacing.

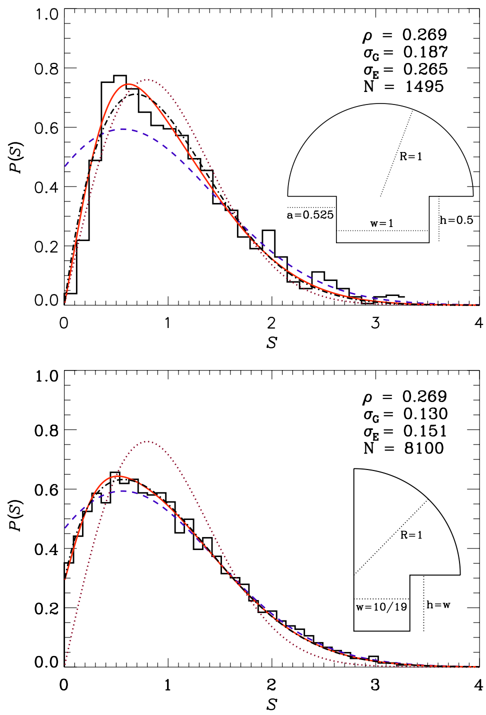

Figure 8. Top: Comparison of experimental data (histogram) with the best-fitting theoretical curves for P^{An}_{DBRN}: full line for the Gaussian model, dash-dotted for the exponential model, dashed for BR (with the same ρ), and dotted for the Wigner distribution. Bottom: comparison of the numerical data (histogram) with the best-fitting theoretical curves for P^{Tn}_{DBRN}: full line for the Gaussian model, dashed-dotted for the exponential model, dashed for BR (with the same ρ) and dotted for the Wigner distribution. σ_G and σ_E are the best-fitting values of σ for the Gaussian and the exponential model, respectively. N_o is the number of objects in the histogram. For other details see text.

For the experimental spectral data, we have employed the usual measuring techniques in the microwave resonators developed by Stöckmann *et al* (see e.g. [2]), and we did so for 21 configurations by varying the geometrical parameter a, cf inset of figure 8, equidistantly around the central value $a = 0.525$ (step $\Delta a = -0.0025$). The parameter ρ for these configurations varies only by about 8% around the central value $\rho = 0.269$. In each configuration, we used the spectral stretches in the interval of 100–300 consecutive levels, and there were quite a few levels missing, so a careful analysis of data was necessary to not involve spectral stretches with missing levels (resonances). The physical scale of the geometry is specified uniquely by noting that in the experiment we used an aluminium cavity with $R = 400$ mm. The thickness of the cavity was 8 mm. The frequency interval was 3.1–5.3 GHz. In both cases, numerical and experimental, we have chosen the examples where we think the statistical significance is the best.

6. Discussion and conclusions

In this paper, we have deviced a random matrix model which describes the distorted Berry–Robnik (BR) level spacing distribution due to tunneling between the regular and chaotic states, denoted by $P^{Tn}_{DBRN}(S)$. We derived a two-level analytic formula for $P^{Tn}_{DBR}(S)$ which well agrees with the results for random matrices of the higher dimension N at not too large value of the coupling parameter σ.

We also developed an analytic two-level model, leading to the level spacing distribution $P_{DBR}^{An}(S)$, for all-to-all level couplings, which applies e.g. in the case of a general perturbation such as the presence of an antenna in a microwave resonator (billiard), and is in excellent agreement with the N-dimensional simulations $P_{DBRN}^{An}(S)$.

While the random matrix model is generally of a high dimension N, we found the analytic two-dimensional random matrix model to be very successful. Its exact integral representation $P_{DBR}^{An}(S)$ for the antenna distorted Berry–Robnik distribution (where all levels are supposed to be coupled) can be expressed analytically in a closed form for the small S and large S regimes. For medium S, an integral representation is given. The same applies to the tunneling distorted Berry–Robnik distribution $P_{DBR}^{Tn}(S)$, where only regular and chaotic levels are coupled by tunneling. Still, an overall good *simple* analytic approximation is lacking. In a forthcoming paper [54], we are going to further improve the present theory and to test its applicability in specific quantum Hamiltonian systems of mixed type.

We have also demonstrated that other than Gaussian models for the couplings (the exponential and box distributions for the off-diagonal matrix elements) agree with the Gaussian model quite well, but nevertheless deviate significantly from the Gaussian model at intermediate values of S. However, they always almost perfectly agree with their two-dimensional approximation (two-level model). The experimental and numerical data for the mushroom billiard can be nicely described by our theoretical models. With increasing energy toward infinity (the semiclassical limit) σ must go to zero (and we find the BR behavior), however not necessarily fast and monotonically. In our case we found that σ goes to zero relatively fast, but not monotonously.

The main signature of a general all-to-all level coupling is the immediate emergence of linear level repulsion, which is a very robust phenomenon described in formulae (5) and (6), and more specifically in (9) and (10). This robustness in general has recently been studied in [56]. Recently, two other attempts to model the tunneling effects have been published in [51] and [39]. The overall picture of level spacings in mixed type systems has been reviewed and discussed in [40]. According to this picture there is at a small S the regime of linear level repulsion, then at larger S a regime of the fractional power-law level repulsion and then the Berry–Robnik tail.

In this paper we have addressed only the tunneling effects, and no localization effects have been treated so far. These latter are probably responsible for the fractional power-law level repulsion, captured by the Brody-like distribution [28]. A 2D random matrix model, as described by the general formula (5), with the singular matrix element distribution function for the a's, the diagonal elements, has been proposed in [56]. Indeed, the Hamilton operators of nearly integrable systems (slightly perturbed integrable systems in the KAM scenario) are quantally represented on the basis of the integrable part as sparsed matrices, which are expected to have such a singular distribution of matrix elements [62]. Research in this direction is under way.

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